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Structural Dynamic Frequency Response Using Combined Direct and Adjoint Reduced-Order Approximations

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A second-order combined approximate-direct/approximate-adjoint method for efficient reduced-order calculation of frequency response in structural dynamics is presented. The method is design oriented. It focuses not only on the response itself (in the form of displacements and, especially, stresses) but also on sensitivities of the response with respect to structural design variables. Comparisons of approximate results obtained by mode-displacement order reduction, reduced-order adjoint solutions based on Ritz vectors, and the second-order approximate method demonstrate the accuracy of the second-order method, and provide lessons that suggest directions for future research.

Nomenclature

$[A]$	=	coefficient matrix for a general linear set of equations
$\{b\}$	=	right-hand side of a general linear set of equations
$[C]$	=	viscous damping matrix
$\{c\}$	=	response recovery vector [Eq. (2)]
$\{F\}$	=	input force vector
$[K]$	=	stiffness matrix
$[M]$	=	mass matrix
Nf	=	number of excitation frequencies
NL	=	number of load cases
NR	=	number of responses
P	=	generic design variable
$\{q\}$	=	generalized displacements
$\{r\}$	=	generalized displacements for the adjoint problem
$\{u\}$	=	displacement vector
$\{x\}$	=	direct solution of a general linear set of equations
y	=	scalar response function [Eqs. (2) and (17)]
δ	=	error entity
$\{\eta\}$	=	adjoint solution vector
$[\Phi]$	=	modal matrix for order reduction of the direct problem
$[\Psi]$	=	reduced-order basis for the adjoint
ω	=	frequency

Subscripts and Superscripts

MA	=	mode acceleration
MD	=	mode displacement
T	=	transpose
$[\Phi]$	=	mode displacement reduced-order matrix
$\Phi\Psi$	=	see Eq. (30)
$[\Psi]$	=	reduced-order matrix for the adjoint problem
$\tilde{}$	=	approximate solution or response function

Introduction

THE problem of mechanical or structural dynamic frequency response optimization of large-scale dynamic systems still presents a significant challenge even with the best of computer power and storage capabilities available today. With mathematical models containing thousands to millions of degrees of freedom, and hundreds, even thousands, of frequencies where response has to be evaluated, this is a computationally intensive problem.[1] Even if hundreds of CPUs are used to evaluate the response at all required excitation frequencies (with each processor handling the detailed dynamic response problem at one frequency only), this is still a formidable problem because of the size of the matrix equations and the computational cost of associated behavior sensitivities required for optimization.

Presented here is a new approach to the problem of design-oriented frequency response of large-scale structural dynamic models based on the finite element displacement formulation. As is well known, model-order reduction in this case can yield inaccurate stress and stress sensitivity information even when displacements are predicted accurately. The new method is especially tailored to produce accurate frequency-dependent stresses and stress sensitivities. The detailed problem in this approach is replaced by a multitude of reduced-order problems (and their reduced-order behavior sensitivities), each small and computationally inexpensive. These reduced-order problems are approximate. But a careful choice of order reduction techniques and the simultaneous utilization of approximate direct as well as approximate adjoint solutions lead to approximation errors, which are second order.

High-accuracy approximations are very important in frequency response optimization studies. They must be capable of capturing the changes in resonant peaks (both frequency and damping of all important modes) and the effect of changes in design on overall changes in transfer functions. The combination of high-accuracy reduced-order approximations and emerging multiprocessor parallel machines (with thousands of cheap processors) promises to lead to practical optimization of large-scale structural dynamic systems in the frequency domain. This work describes an exploratory effort to evaluate the accuracy of second-order reduced-order approximations for frequency-dependent stresses in structural dynamic systems.

Detailed and Modally Reduced-Order Models

Linear dynamic equations for a structural dynamic system in the frequency domain are usually written as

$$[-\omega^2[M] + j\omega[C] + [K]]\{u(j\omega)\} = \{F(j\omega)\} \tag{1}$$

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A single scalar response can be obtained from the vector of responses by

$$y(j\omega) = \{c\}^T\{u(j\omega)\} = \{\eta(j\omega)\}^T\{F(j\omega)\} \tag{2}$$

Here the detailed adjoint problem, for the case where the matrices $[M]$, $[K]$ are symmetric and the vector $\{c\}$ is frequency-independent, is

$$[-\omega^2[M] + j\omega[C]^T + [K]]\{\eta(j\omega)\} = \{c\} \tag{3}$$

We use the term "detailed model" here to mean large number of degrees of freedom corresponding to a detailed full-order finite element model.

Order Reduction Using Reduced Modal Bases

In the mode-displacement (MD) method a reduced basis matrix $[\Phi]$ is used to reduce the order of the direct problem as follows:

$$\{u_{\text{MD}}(j\omega\} = [\Phi]\{q(j\omega)\} \tag{4}$$

$$\left[-\omega^2[M_\Phi] + j\omega[C_\Phi] + [K_\Phi]\right]\{q(j\omega)\} = \{F_\Phi(j\omega)\} \tag{5}$$

The vector $\{q(j\omega)\}$ contains generalized displacements, while the generalized mass, damping, and stiffness matrices, respectively, are

$$[M_\Phi] = [\Phi]^T[M][\Phi] \tag{6}$$

$$[C_\Phi] = [\Phi]^T[C][\Phi] \tag{7}$$

$$[K_\Phi] = [\Phi]^T[K][\Phi] \tag{8}$$

The generalized input force vector is given by

$$\{F_\Phi(j\omega)\} = [\Phi]^T\{F(j\omega)\} \tag{9}$$

In a similar manner a set of reduced-order adjoint frequency response equations can be obtained by using some reduced basis matrix $[\Psi]$. The generalized adjoint displacements are given by the vector $\{r(j\omega)\}$, where

$$\{\eta(j\omega\} = [\Psi]\{r(j\omega)\} \tag{10}$$

The reduced-order adjoint equations are

$$\left[-\omega^2[M_\Psi] + j\omega[C_\Psi]^T + [K_\Psi]\right]\{r(j\omega)\} = \{c_\Psi\} \tag{11}$$

The generalized mass, damping, and stiffness matrices, respectively, with respect to $[\Psi]$ are

$$[M_\Psi] = [\Psi]^T[M][\Psi] \tag{12}$$

$$[C_\Psi] = [\Psi]^T[C][\Psi] \tag{13}$$

$$[K_\Psi] = [\Psi]^T[K][\Psi] \tag{14}$$

and the corresponding generalized $\{c\}$ vector is given by

$$\{c_\Psi\} = [\Psi]^T\{c\} \tag{15}$$

The selection of reduced bases $[\Phi]$ and $[\Psi]$ is not straightforward and depends on the spatial distribution of excitation forces, the weights used to determine particular responses, as well as the frequency of excitation. Utilization of natural modes of the structure, "fictitious mass" modes, Ritz vectors, or alternative subcomponent modes[2–8] has been studied extensively over the years. Order reduction of both direct and adjoint methods in static and transient structural dynamics is discussed in Refs. 9 and 10.

Second-Order Approximation Based on Use of Approximate Direct and Adjoint Solutions

Let a direct solution $\{\boldsymbol{x}\}$ of a linear problem be obtained by solving the matrix equation

$$[A]\{x\} = \{b\} \tag{16}$$

A particular scalar response y can be found using either the direct or adjoint solution:

$$y = \{c\}^T\{x\} = \{c\}^T[A]^{-1}\{b\} = \{\eta\}^T\{b\} \tag{17}$$

where the adjoint problem is given by

$$[A]^T\{\eta\} = \{c\} \tag{18}$$

In optimization studies of large-scale problems, where solutions to the detailed full-order equations are computationally expensive, it is important to utilize approximate, computationally inexpensive solutions for the repetitive analyses and sensitivity analyses involved. If such approximate solutions obtained from reduced-order or simplified models are denoted $\{\tilde{x}\}$, $\{\tilde{\eta}\}$, $\tilde{y}$, then the corresponding errors can be defined by

$$\{\delta x\} = \{x\} - \{\tilde{x}\}, \qquad \{\delta\eta\} = \{\eta\} - \{\tilde{\eta}\}, \qquad \delta y = y - \tilde{y} \tag{19}$$

Both direct and adjoint approximate solutions can now be used simultaneously[9–11] to create a new approximation for the scalar response y, with an error term that depends on errors in $\{x\}$ and $\{\eta\}$ to second order (see Appendix). The new approximation of y is constructed using the approximate direct and adjoint solutions and the exact full-order matrix $[A]$ and vectors $\{b\}$, $\{c\}$:

$$\tilde{y} = \{c\}^T\{\tilde{x}\} + \{\tilde{\eta}\}^T\{b\} - \{\tilde{\eta}\}^T[A]\{\tilde{x}\} \tag{20}$$

$$\{\tilde{x}\} = \{x\} - \{\delta x\}, \qquad \{\tilde{\eta}\} = \{\eta\} - \{\delta\eta\}, \qquad \tilde{y} = y - \delta y \tag{21}$$

Substitution of Eq. (21) into Eq. (20), using Eqs. (16–18), and collecting terms will lead to an expression for the error in y:

$$\delta y = -\delta\eta^T A \delta x \tag{22}$$

An alternative second-order approximation, using the approximate $\{\tilde{x}\}$, $\{\tilde{\eta}\}$ and exact $[A]$, $\{b\}$, $\{c\}$, can also be constructed:

$$\tilde{y} = \frac{(\{c\}^T\{\tilde{x}\})(\{\tilde{\eta}\}^T\{b\})}{\{\tilde{\eta}\}^T[A]\{\tilde{x}\}} \tag{23}$$

where by substituting Eq. (21) into a variation of Eq. (23)

$$\{\tilde{\eta}\}^T[A]\{\tilde{x}\}\tilde{y} = (\{c\}^T\{\tilde{x}\})(\{\tilde{\eta}\}^T\{b\}) = \{\tilde{\eta}\}^T\{b\}\{c\}^T\{\tilde{x}\}$$

and using Eqs. (16–18) it can be shown that δy is second order in $\{\delta x\}$ and $\{\delta\eta\}$. Note the similarity of the quotient in Eq. (23) to a Rayleigh quotient[12,13] for generalized eigenvalue problems:

$$\tilde{y} = \frac{(\{c\}^T\{\tilde{x}\})(\{\tilde{\eta}\}^T\{b\})}{\{\tilde{\eta}\}^T[A]\{\tilde{x}\}} = \frac{\{\tilde{\eta}\}^T\lfloor\{b\}\{c\}^T\rfloor\{\tilde{x}\}}{\{\tilde{\eta}\}^T[A]\{\tilde{x}\}} \tag{24}$$

The structure of the Rayleigh quotient and its well-known second-order accuracy can be used to guide the derivation of the approximate equation (23).

Frequency Response of Structural Dynamic Systems

For a structural dynamic system defined by its mass, damping, and stiffness matrices [Eq. (1)], the frequency response of some output y [(Eq. (2)] as a result of a frequency-dependent loading force vector $\{F(j\omega)\}$ is calculated numerically by solving either Eq. (1) or (3) for a set of excitation frequencies $(\omega_1, \omega_2, \ldots, \omega_{Nf})$. The selected frequencies must cover the frequency range of interest and have a fine enough mesh around resonant frequencies to capture resonant behavior. Equation (1) has to be solved at Nf frequencies for each of NL load cases (right-hand sides). Equation (3), if used, must be solved at Nf frequencies for each of NR responses (using NR right-hand-side vectors {c}). With full-order models for large-scale systems, this is an intensive computational task. If design optimization is attempted, where both the responses and their sensitivities have to be found repeatedly for varying mass, damping and stiffness matrices, the required computational effort can be formidable.

To overcome this problem, it is quite common in structural dynamics to calculate frequency response of large-scale systems using reduced-order equations [Eqs. (4) and (5)]. Whether normal modes are used, or Ritz vectors, or combinations of Ritz vectors and normal modes, or "fictitious mass" modes, the approximations of displacements, velocities, and natural frequencies can be quite accurate, provided that the reduced basis used covers the span of frequencies of interest and takes the spatial distribution of input forces into account.[2–10] This is not the case, however, when stresses in the structure are sought, or when behavior sensitivities of displacements and stresses with respect to structural design variables are needed.[9,10,14,15] In the case of structural optimization, when the structure changes during the design optimization process, and matrices $[M]$, $[C]$, and $[K]$ vary, it is very expensive computationally to create new reduced bases for order reduction with every change in the structure. A method of "fixed modes" is usually preferred[14,15] in which a single reduced basis generated at some reference configuration serves to reduce the order of the problem for a large number of different designs that evolve during the search for the optimum. Such fixed-modes order reduction in Eqs. (4) and (5) might lead to larger and larger errors as the design wanders away from its original reference point.

As an extension of the work reported in Refs. 9 and 10, a second-order approximation for design optimization involving frequency response of structural dynamic systems and based on a fixed-modes approach will be presented in the following sections.

Second-Order Approximation Method for Structural Dynamic Response of Linear Systems

In one variant of a second-order approximate method for frequency-response calculations, the reduced-order approximate direct and adjoint equations [Eqs. (5) and (11)] are solved for any new structure over the set of frequencies $(\omega_1, \omega_2, \ldots, \omega_{Nf})$. The reduced bases $[\Phi]$ and $[\Psi]$ are created first for a properly selected reference structure and then held fixed for all subsequent modified structures. The solution of Eqs. (5) can be quite efficient if the low-order matrices $[-\omega^2[M_\Phi] + j\omega[C_\Phi] + [K_\Phi]]$ are L-U decomposed at the frequencies ω_i and then used to solve for generalized displacements [Eq. (4)] for each load case. Similarly, the matrices $[-\omega^2[M_\Psi] + j\omega[C_\Psi]^T + [K_\Psi]]$ are L-U decomposed at each frequency ω_i and the decompositions then used to find generalized adjoint responses [Eq. (11)] for different response functions y (and corresponding $\{c\}$ vectors). With approximate solutions

$$\{\tilde{u}(j\omega_i\} = [\Phi]\{q(j\omega_i)\} \tag{25}$$

$$\{\tilde{\eta}(j\omega_i\} = [\Psi]\{r(j\omega_i)\} \tag{26}$$

Second-order approximations of the dynamic response k as a result of load case excitation l can be constructed using either Eq. (20) or (23). In the case of Eq. (20),

$$\begin{aligned}\tilde{y}_{k,l}(j\omega_i) &= \{c_k\}^T\{\tilde{u}_l(j\omega_i)\} + \{\tilde{\eta}_k(j\omega_i)\}^T\{F_l(j\omega_i)\}\\ &\quad - \{\tilde{\eta}_k(j\omega_i)\}^T\big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\{\tilde{u}_l(j\omega_i)\}\end{aligned} \tag{27}$$

In Eq. (27) the matrices $[M], [C], [K]$ and the vectors $\{F_l(j\omega_i)\}, \{c_k\}$ are the full-order matrices and vectors for the modified structure.

Alternatively, based on Eq. (23), the approximate second-order response k to excitation force vector l is expressed as

$$\tilde{y}_{k,l}(j\omega_i) = \frac{\big(\{c_k\}^T\{\tilde{u}_l(j\omega_i)\}\big)\big(\{\tilde{\eta}_k(j\omega_i)\}^T\{F_l(j\omega_i)\}\big)}{\{\tilde{\eta}_k(j\omega_i)\}^T\big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\{\tilde{u}_l(j\omega_i)\}} \tag{28}$$

With substitution of Eqs. (25) and (26) into Eq. (27), we get for the first approximation

$$\begin{aligned}\tilde{y}_{k,l}(j\omega_i) &= \{c_k\}^T[\Phi]\{q_l(j\omega_i)\} + \{r_k(j\omega_i)\}^T[\Psi]^T\{F_l(j\omega_i)\}\\ &\quad - \{r_k(j\omega_i)\}^T\big[-\omega_i^2[M_{\Psi\Phi}] + j\omega_i[C_{\Psi\Phi}] + [K_{\Psi\Phi}]\big]\{q_l(j\omega_i)\}\end{aligned} \tag{29}$$

where

$$[M]_{\Psi\Phi} = [\Psi]^T[M][\Phi], \qquad [C]_{\Psi\Phi} = [\Psi]^T[C][\Phi]$$
$$[K]_{\Psi\Phi} = [\Psi]^T[K][\Phi] \tag{30}$$

Expressions for the second approximation [Eq. (28)] in terms of the generalized reduced-order direct and adjoint displacement vectors $\{q_l\}$, $\{r_k\}$ can be obtained in a similar way.

Sensitivities

Differentiation of Eq. (27) with respect to a design variable p leads to

$$\begin{aligned}\frac{\partial\tilde{y}_{k,l}(j\omega_i)}{\partial p} &= \frac{\partial\{c_k\}^T}{\partial p}\{\tilde{u}_l(j\omega_i)\} + \{c_k\}^T\frac{\partial\{\tilde{u}_l(j\omega_i)\}}{\partial p}\\ &\quad + \frac{\partial\{\tilde{\eta}_k(j\omega_i)\}^T}{\partial p}\{F_l(j\omega_i)\} + \{\tilde{\eta}_k(j\omega_i)\}^T\frac{\partial\{F_l(j\omega_i)\}}{\partial p}\\ &\quad - \frac{\partial\{\tilde{\eta}_k(j\omega_i)\}^T}{\partial p}\big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\{\tilde{u}_l(j\omega_i)\}\\ &\quad - \{\tilde{\eta}_k(j\omega_i)\}^T\big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\frac{\partial\{\tilde{u}_l(j\omega_i)\}}{\partial p}\\ &\quad - \{\tilde{\eta}_k(j\omega_i)\}^T\left[-\omega_i^2\frac{\partial[M]}{\partial p} + j\omega_i\frac{\partial[C]}{\partial p} + \frac{\partial[K]}{\partial p}\right]\{\tilde{u}_l(j\omega_i)\}\end{aligned} \tag{31}$$

After collection of terms,

$$\begin{aligned}\frac{\partial\tilde{y}_{k,l}(j\omega_i)}{\partial p} &= \frac{\partial\{c_k\}^T}{\partial p}\{\tilde{u}_l(j\omega_i)\} + \{\tilde{\eta}_k(j\omega_i)\}^T\frac{\partial\{F_l(j\omega_i)\}}{\partial p}\\ &\quad - \{\tilde{\eta}_k(j\omega_i)\}^T\left[-\omega_i^2\frac{\partial[M]}{\partial p} + j\omega_i\frac{\partial[C]}{\partial p} + \frac{\partial[K]}{\partial p}\right]\{\tilde{u}_l(j\omega_i)\}\\ &\quad + \frac{\partial\{\tilde{\eta}_k(j\omega_i)\}^T}{\partial p}\big\langle\{F_l(j\omega_i)\} - \big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\\ &\quad \times\{\tilde{u}_l(j\omega_i)\}\big\rangle + \big\langle\{c_k\}^T - \{\tilde{\eta}_k(j\omega_i)\}^T\big[-\omega_i^2[M]\\ &\quad + j\omega_i[C] + [K]\big]\big\rangle\frac{\partial\{\tilde{u}_l(j\omega_i)\}}{\partial p}\end{aligned} \tag{32}$$

We notice that the expressions

$$\big\langle\{F_l(j\omega_i)\} - \big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\{\tilde{u}_l(j\omega_i)\}\big\rangle \tag{33}$$

$$\big\langle\{c_k\}^T - \{\tilde{\eta}_k(j\omega_i)\}^T\big[-\omega_i^2[M] + j\omega_i[C] + [K]\big]\big\rangle \tag{34}$$

actually correspond to Eqs. (1) and (3), with the exact direct and adjoint solutions replaced by the approximate ones. If these approximations render expressions (33) and (34) very small, the terms containing them in Eq. (32) can be neglected, and the approximate sensitivity equations can be simplified considerably. With the additional simplifying assumptions that the input forces and $\{c\}$ vectors do not depend on design variables, the approximate reduced-order sensitivity equation can now be written as

$$\frac{\partial \tilde{y}_{k,l}(j\omega_i)}{\partial p} \approx -\{\tilde{\eta}_k(j\omega_i)\}^T\left[-\omega_i^2\frac{\partial[M]}{\partial p}+j\omega_i\frac{\partial[C]}{\partial p}+\frac{\partial[K]}{\partial p}\right]\{\tilde{u}_l(j\omega_i)\} \tag{35}$$

The sensitivities of the approximate direct and adjoint solutions are obtained from the sensitivities of Eqs. (4) and (5) and (10) and (11):

$$\left\{\frac{\partial\tilde{u}(j\omega)}{\partial p}\right\}=[\Phi]\left\{\frac{\partial q(j\omega)}{\partial p}\right\} \tag{36}$$

where

$$\left[-\omega^2[M_\Phi]+j\omega[C_\Phi]+[K_\Phi]\right]\left\{\frac{\partial q(j\omega)}{\partial p}\right\} = -\left[-\omega^2\frac{\partial[M_\Phi]}{\partial p}+j\omega\frac{[\partial C_\Phi]}{\partial p}+\frac{\partial[K_\Phi]}{\partial p}\right]\{q(j\omega)\} \tag{37}$$

and

$$\left\{\frac{\partial\tilde{\eta}(j\omega)}{\partial p}\right\}=[\Psi]\left\{\frac{\partial r(j\omega)}{\partial p}\right\} \tag{38}$$

where

$$\left[-\omega^2[M_\Psi]+j\omega[C_\Psi]^T+[K_\Psi]\right]\left\{\frac{\partial r(j\omega)}{\partial p}\right\} = -\left[-\omega^2\frac{\partial[M_\Psi]}{\partial p}+j\omega\frac{\partial[C_\Psi]^T}{\partial p}+\frac{\partial[K_\Psi]}{\partial p}\right]\{r(j\omega)\} \tag{39}$$

Second-Order Approximation for Frequency-Response Mode Acceleration: Using Direct and Adjoint Reduced-Order Solutions

For the second-order approximations just presented, it was necessary to obtain a reduced-order frequency-dependent MD direct solution [Eqs. (4) and (5)] as well as a frequency-dependent reduced-order adjoint solution [Eqs. (10) and (11)]. In an extension of the work reported in Refs. 9 and 10, a reduced-order variant of the mode-acceleration (MA) method[16,17] can be developed for the second-order approximation. Such a method will be particularly important when some of the desired outputs y are stresses or any other spatial derivatives of the unknown displacements.[9,10] The MA method is based on a static solution of the full-order static equations with an approximate MD dynamic (frequency-dependent) right-hand side:

$$[K]\{u_{\mathrm{MA}}(j\omega)\}=\{F_{\mathrm{MA}}(j\omega\} \tag{40}$$

where the frequency-dependent ("dynamic") right-hand side is

$$\{F_{\mathrm{MA}}(j\omega)\}=\{F(j\omega\}+\omega^2[M]\{u_{\mathrm{MD}}(j\omega)\}-j\omega[C]\{u_{\mathrm{MD}}(j\omega)\} \tag{41}$$

Dynamic displacement solutions of the mode-displacement equations [Eqs. (4) and (5)] are denoted MD. The assumption is that MD solutions are accurate enough for approximating the full-order right-hand side of Eq. (40), but that for stress information the full-order stiffness matrix must be used. Equations (40) and (41) represent a quasi-static problem. When a response $y(j\omega)=\{c\}^T\{u(j\omega)\}$ is sought, it can be obtained from the adjoint of the quasi-static problem by

$$y_{\mathrm{MA}}(j\omega)=\{c\}^T\{u_{\mathrm{MA}}(j\omega)\}=\{\eta_{\mathrm{MA}}\}^T\{F_{\mathrm{MA}}(j\omega)\} \tag{42}$$

where (for the case of a nonsingular stiffness matrix)

$$[K]\{\eta_{\mathrm{MA}}\}=\{c\} \tag{43}$$

Both Eqs. (40) and (43) can be solved approximately, in reduced order, by using reduced-order bases

$$\{\tilde{u}_{\mathrm{MA}}(j\omega\}=[\Phi]\{q_{\mathrm{MA}}(j\omega)\} \tag{44}$$

$$\{\tilde{\eta}_{\mathrm{MA}}\}=[\Psi]\{r_{MA}\} \tag{45}$$

Substitution of Eq. (44) into Eq. (40) and using Eqs. (4) and (5), it immediately becomes evident that

$$\{q_{\mathrm{MA}}\}=\{q\}=\{q_{\mathrm{MD}}\} \tag{46}$$

That is, the approximate direct solution used for the mode-acceleration method is just the mode-displacement reduced-order solution. The reduced-order adjoint for the MA method is given by

$$[\Psi]^T[K][\Psi]\{r_{\mathrm{MA}}\}=[\Psi]^T\{c\} \tag{47}$$

Compared with the dynamic adjoint solution of Eqs. (10) and (11), here the system's matrix on the left-hand side is not frequency dependent, and it is created and decomposed only once. A second-order approximation can now be constructed as follows:

$$\tilde{y}(j\omega)=\{c\}^T\{u_{\mathrm{MD}}(j\omega)\}+\{\tilde{\eta}_{\mathrm{MA}}\}^T\{F_{\mathrm{MA}}(j\omega\} - \{\tilde{\eta}_{\mathrm{MA}}\}^T[K]\{u_{\mathrm{MD}}(j\omega)\} \tag{48}$$

Or, in a quotient form,

$$\tilde{y}(j\omega)=\frac{\left(\{c\}^T\{u_{\mathrm{MD}}(j\omega)\}\right)\left(\{\tilde{\eta}_{\mathrm{MA}}\}^T\{F_{\mathrm{MA}}(j\omega\}\right)}{\{\tilde{\eta}_{\mathrm{MA}}\}^T[K]\{u_{\mathrm{MD}}(j\omega)\}} \tag{49}$$

Sensitivities, in a way similar to the techniques in Refs. 9 and 10, can be obtained by direct differentiation of Eqs. (48) and (49). In the case of Eq. (48),

$$\frac{\partial\tilde{y}(j\omega)}{\partial p}=\frac{\partial\{c\}^T}{\partial p}\{u_{\mathrm{MD}}(j\omega)\}+\{\tilde{\eta}_{\mathrm{MA}}\}^T\frac{\partial\{F_{\mathrm{MA}}(j\omega\}}{\partial p} +\left(\{c\}^T-\{\tilde{\eta}_{\mathrm{MA}}\}^T[K]\right)\frac{\partial\{u_{\mathrm{MD}}(j\omega)\}}{\partial p}+\frac{\partial\{\tilde{\eta}_{\mathrm{MA}}\}^T}{\partial p}\{\{F_{\mathrm{MA}}(j\omega)\} -[K]\{u_{\mathrm{MD}}(j\omega)\}\}-\{\tilde{\eta}_{\mathrm{MA}}\}^T\frac{\partial[K]}{\partial p}\{u_{\mathrm{MD}}(j\omega)\} \tag{50}$$

with

$$\frac{\partial\{F_{\mathrm{MA}}(j\omega)\}}{\partial p}=\frac{\partial\left(\{F(j\omega\}+[\omega^2[M]-j\omega[C]]\{u_{\mathrm{MD}}(j\omega)\}\right)}{\partial p} \tag{51}$$

and $\partial\{u_{\mathrm{MD}}(j\omega)\}/\partial p$ obtained from the mode-displacement equation sensitivities [Eqs. (36) and (37)].

Again, if the $\{c\}$ and $\{F\}$ vectors do not depend on design variables, and if the error caused by using the approximate $\{u_{\mathrm{MD}}\}$ and $\{\tilde{\eta}_{\mathrm{MA}}\}$ into Eqs. (40) and (43) are small, then

$$\frac{\partial\tilde{y}(j\omega)}{\partial p}=\{\tilde{\eta}_{\mathrm{MA}}\}^T\frac{\partial\{F_{\mathrm{MA}}(j\omega\}}{\partial p}-\{\tilde{\eta}_{\mathrm{MA}}\}^T\frac{\partial[K]}{\partial p}\{u_{\mathrm{MD}}(j\omega)\} \tag{52}$$

where [Eqs. (36) and (37)]

$$\frac{\partial\{F_{\mathrm{MA}}(j\omega)\}}{\partial p}=[\omega^2[M]-j\omega[C]]\frac{\partial\{u_{\mathrm{MD}}(j\omega)\}}{\partial p} +\left[\omega^2\frac{\partial[M]}{\partial p}-j\omega\frac{\partial[C]}{\partial p}\right]\{u_{\mathrm{MD}}(j\omega)\} \tag{53}$$

Selection of Reduced Bases [Φ], [Ψ]

Real Valued [Φ], [Ψ]

As discussed thoroughly in Refs. 9 and 10, there is quite a variety of reduced basis choices for both the direct and adjoint problems. Important choices of [Φ] include the low-frequency set of natural mode shapes of a reference structure from which the optimized design evolves, Ritz vectors derived from the spatial distribution of input forces,[4–6] fictitious-mass modes of a reference structure,[8,18–20] or a combination of Ritz vectors and reference mode shapes.[7]

The emphasis in Refs. 9 and 10 was on obtaining accurate stress information. It was found that a successful [Ψ] for both the adjoint static problem and mode-acceleration problem was constructed by taking (as columns in the [Ψ] matrix) static responses to unit loads at the degrees of freedom at the nodes of the element containing the desired stress. If a displacement response at a node is sought, then responses to unit loads at the nodes of the elements attached to that node will serve successfully as columns in [Ψ]. The performance of a single vector [Ψ] matrix, corresponding to the adjoint loading $\{c\}$ in the reference configuration, was found to be inferior, although the reduction of the adjoint problem [Eqs. (11) and (47)] to a single-degree-of-freedom equation is quite tempting, and further exploration of this possibility is desired.

Frequency-Dependent Complex Valued [Φ ($j\omega$)], [Ψ($j\omega$)]

In addition to the [Φ], [Ψ] selection alternatives just discussed, the frequency-dependent problems addressed here allow the selection of frequency-dependent reduced bases $[\Phi(j\omega)]$, $[\Psi(j\omega)]$. The approach explored here is still a fixed-modes approach. That is, the matrices $[\Phi(j\omega)]$, $[\Psi(j\omega)]$ do not change with changes in design variables. There might, however, be different reduced base matrices for different frequency bands. For example, the columns in the [Φ], [Ψ] matrices used to create the reduced-order equations at a frequency ω_i might include the full-order solutions of the reference (at the base design) problem at ω_i plus a collection of such reference responses at adjacent frequencies. The reference response vectors to be included can, thus, cover a frequency band that is expected to contribute most to the response. Note, however, that frequency-dependent reduced bases here are used in the context of a quasi-steady mode-acceleration approach for which the adjoint solution does not depend on frequency.

Test Case

The structural dynamic system used in the studies reported in this paper is an all-aluminum wing model (Fig. 1; Refs. 21 and 22), modeled by truss and membrane finite elements. The wing is loaded by a concentrated 500-lb (2224.49-N) force applied at 70% span and 60% chord from the leading edge.

The finite element model used has 300 degrees of freedom. A structural damping coefficient of $g = 0.04$ is assumed. The usage of a structural (and not viscous) damping model makes it necessary to modify the structural dynamic equations slightly. The viscous-damping force terms $j\omega[C]\{u\}$ and $j\omega[C]^T\{\eta\}$, in the direct and adjoint equations, respectively, have to be replaced by $jg[K]\{u\}$ and $jg[K]\{\eta\}$.

Stress, for the cases described here, is calculated at the upper cap of the front spar between the second and third ribs. Displacement is calculated at the rear wing-tip node in the z direction (perpendicular to the plane of the wing). Mass loading of the wing is such that the four lowest natural frequencies of the base (reference) wing are 24.427, 42.847, 70.834, and 109.173 rad/s. Stress and displacement

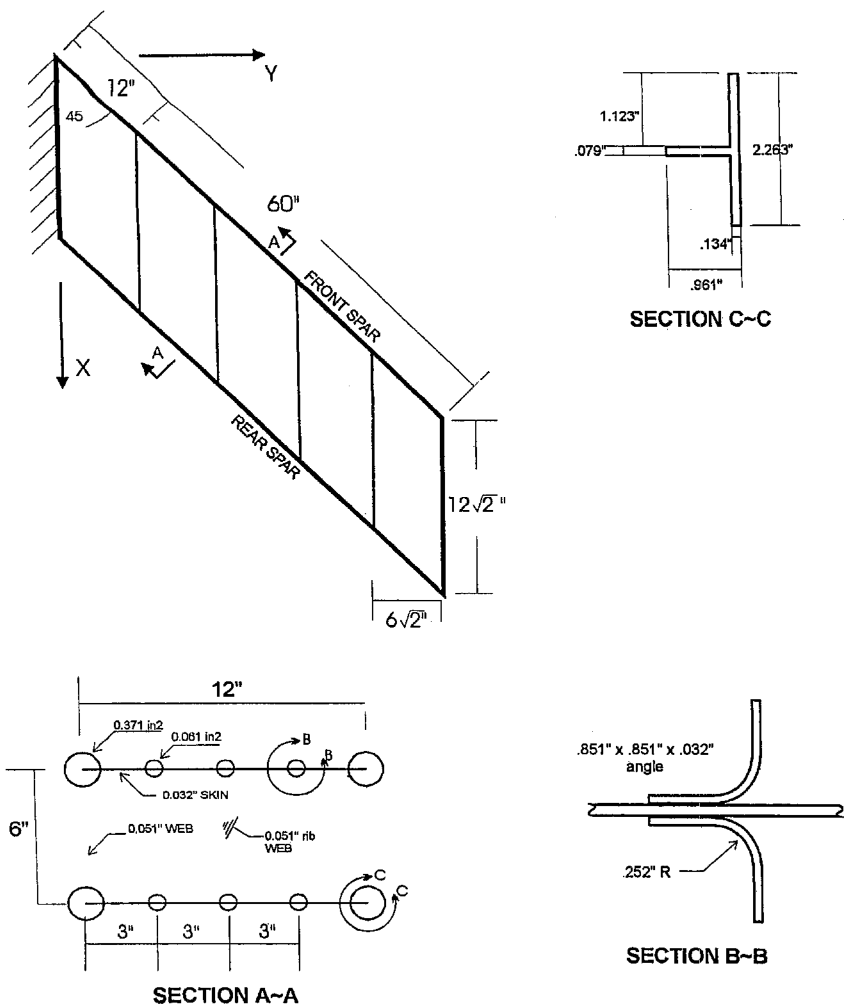

Fig. 1 Swept-back wing model.[9,10,21,22]

computations are carried out for a modified wing, in which the cross-sectional area of a group of 10 rod elements, between the root and first rib, is reduced from the base (reference) value by 20%.

For the order reduction of the direct problem [Eqs. (4) and (5)], a fixed matrix $[\Phi]$ is used, consisting of the six low-frequency mode shapes of the reference (base) structure. This reduced basis, following common practices in structural dynamics, is used throughout the range of excitation frequencies. The number of modes used has to ensure that the corresponding natural frequencies cover the frequency range of interest.

For the order reduction of the adjoint problems [Eqs. (10) and (11)], two types of reduced base matrices $[\Psi]$ are used. In the first type a set of fixed Ritz vectors are used as follows. For the stress response case the Ritz vectors are the full-order adjoint static solutions of the base (reference) structure as a result of unit inputs at each of the degrees of freedom of the element in which the stress is calculated.[9,10] For a three-dimensional truss element this means six degrees of freedom and corresponding six Ritz vectors. For the three-dimensional quadrilateral membrane element, 12 Ritz vectors are generated for the adjoint approximation. For the displacement response, the Ritz vectors are solutions of the static full-order problem as a result of unit force inputs at the degrees of freedom of the node at which displacement is calculated. In our case there are six Ritz vectors for the adjoint solutions corresponding to stress in a truss element and three Ritz vectors (corresponding to x, y, z) inputs at the wing-tip rear node. The second type of reduced basis matrix $[\Psi]$ is frequency dependent. It is made of solutions to the full-order dynamic adjoint problem [Eq. (3)] at different excitation frequencies, with the same kind of right-hand-side inputs as in the static case. Knowing the natural frequencies of the base structure, the frequency range of excitation was divided into five bands, and for frequencies in each of the bands a different $[\Psi]$ matrix was used.

The five intervals were 1) from 0 to 19 rad/s, 2) from 20 to 26 rad/s, 3) from 27 to 48 rad/s, 4) from 50 to 100 rad/s, and 5) from 102 to 116 rad/s. Adjoint Ritz vectors for each interval were calculated as follows: interval 1, static Ritz vectors at 0 rad/s; interval 2, Ritz vectors calculated at 24.427 rad/s; interval 3, Ritz vectors calculated at 42.847 rad/s; interval 4, Ritz vectors calculated at 70.834 rad/s; and interval 5, Ritz vectors calculated at 109.173 rad/s.

The Ritz vectors for intervals 2–5 were calculated at the natural frequencies of the reference (base) structure. It is, of course, expected that the natural frequencies of the modified structure will wander away from the frequencies of the base (reference) structure. The frequency range can be divided into more intervals, but this means additional computational costs caused by an increased number of full-order adjoint solutions of the base structure.

For sensitivity accuracy studies stress sensitivity is calculated for the same truss element used for the stress approximation, and the stress derivative is taken with respect to the area of one of the adjacent elements. Frequency response was calculated at 156 excitation frequencies from 0 to 116 rad/s.

Results

Errors in the approximate stress as functions of excitation frequency are presented in Fig. 2 for the case of fixed adjoint Ritz vectors obtained from a static adjoint solution of the base structure and Fig. 3 for the case of frequency-dependent adjoint Ritz vectors. The frequency-dependent stresses are complex numbers, and only the errors in modulus are presented here.

The figures compare approximate results obtained from a standard mode displacement approximation (with the six low-frequency mode shapes of the base structure), a reduced-order adjoint method (with either fixed or frequency-dependent Ritz vectors), and the second-order approximation. The significant errors of the reduced-order adjoint method, whether fixed or frequency-dependent Ritz vectors are used, are evident. Also evident is the significant improvement in accuracy of the second-order approximation over the whole frequency range when frequency-dependent Ritz vectors are used for the adjoint.

Overall, the performance of the second-order approximation with six mode shapes and six frequency-dependent Ritz vectors is excellent. A 300×300 problem at each excitation frequency can, thus, be replaced by two 6×6 problems: the approximate-direct and approximate-adjoint problems.

Stress sensitivity results are presented in Figs. 4 and 5. Figure 4 is based on fixed static adjoint Ritz vectors, whereas Fig. 5 is based on five sets of Ritz vectors (frequency dependent) evaluated as described in the preceding section.

Note the improved accuracy of the second-order method when used with frequency-dependent adjoint Ritz vectors. The sensitivity errors of the reduced-order adjoint approximations are very large. The reduced-order MD sensitivities are better, but they have large errors around the first and second natural frequencies. The second-order approximation leads to much smaller errors around the first and

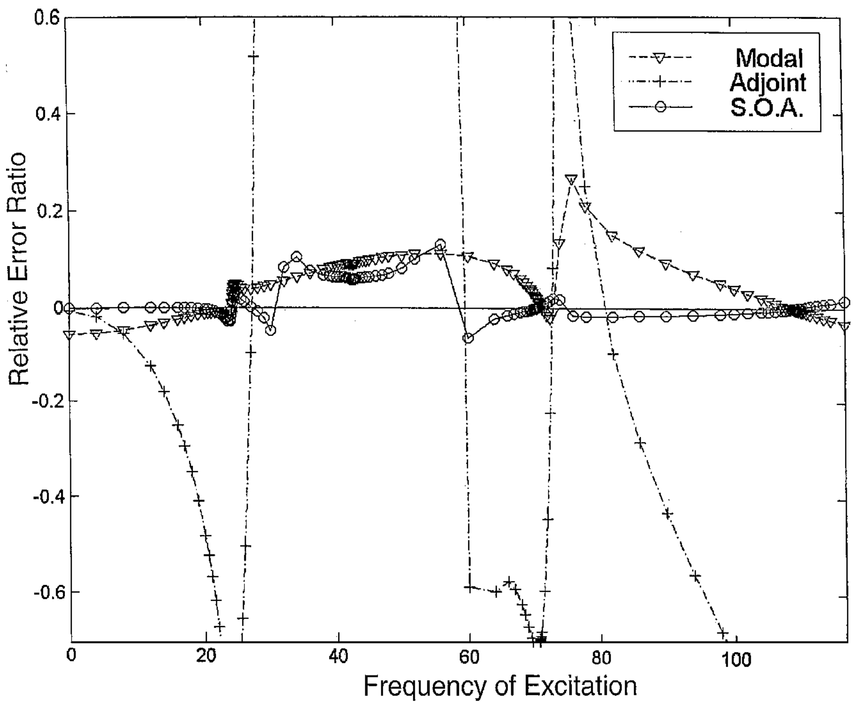

Fig. 2 Stress modulus error as function of excitation frequency (rad/s) using six low-frequency modes of the base structure for the direct problem and six fixed adjoint static Ritz vectors.

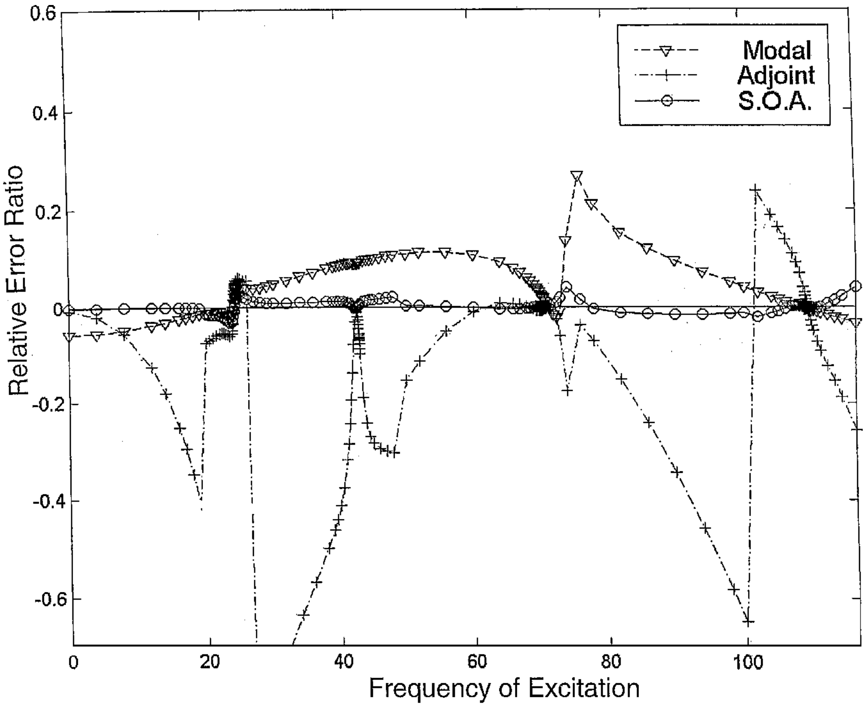

Fig. 3 Stress modulus error as function of excitation frequency (rad/s) using six low-frequency modes of the base structure for the direct problem and six frequency-dependent adjoint Ritz vectors.

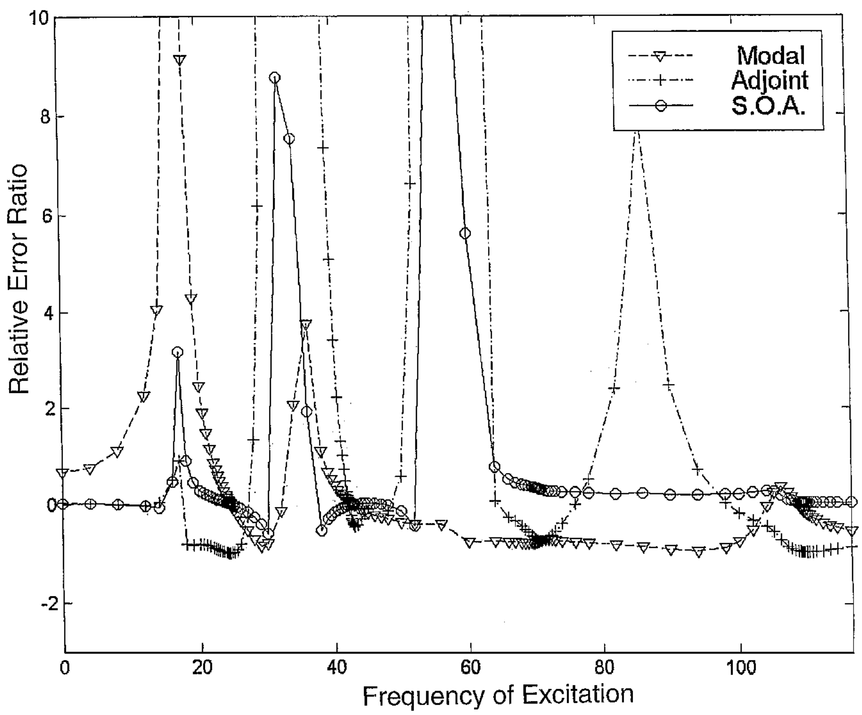

Fig. 4 Stress-sensitivity modulus error as function of excitation frequency (rad/s) using six low-frequency modes of the base structure for the direct problem and six fixed adjoint static Ritz vectors.

second natural frequencies and shows a remarkable capacity to improve accuracy compared to both approximate direct MD problem and approximate Ritz-vector-based adjoint. The largest stress sensitivity error is at about 19 rad/s, the frequency at which frequency-dependent Ritz vectors in $[\Psi]$ are switched from those evaluated at zero frequency to those evaluated at 24.427 rad/s. A more elaborate frequency dependence of the adjoint Ritz vectors should be investigated in future studies in an effort to obtain high accuracy of stress sensitivities over the whole frequency range. Possible improvements include division of the frequency range into more frequency intervals or expansion of the reduced basis in $[\Psi]$ to include Ritz vectors from adjacent frequency intervals. Combinations of modes and Ritz vectors for the direct problem[7] should also be considered. The results presented here also suggest that more research is needed to understand the poor accuracy of the approximate reduced-order adjoint solutions and to develop better reduced-order adjoint approximations. Clearly, more research is required, and additional reduced-order bases need to be explored in the search for efficient low-order approximate frequency response analysis and sensitivity calculations.

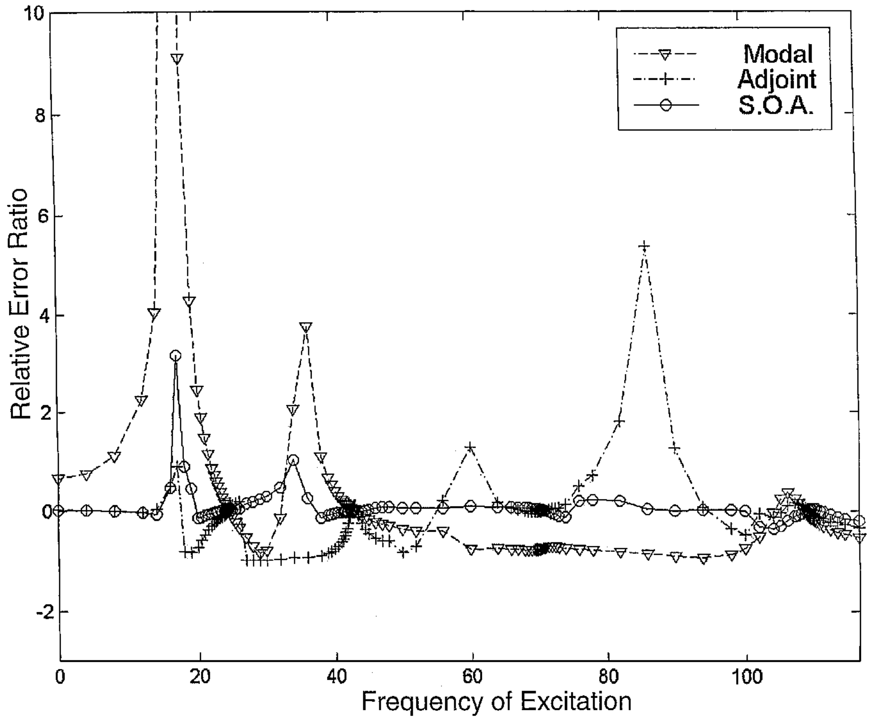

Fig. 5 Stress-sensitivity modulus error as function of excitation frequency (rad/s) using six low-frequency modes of the base structure for the direct problem and six frequency-dependent adjoint static Ritz vectors

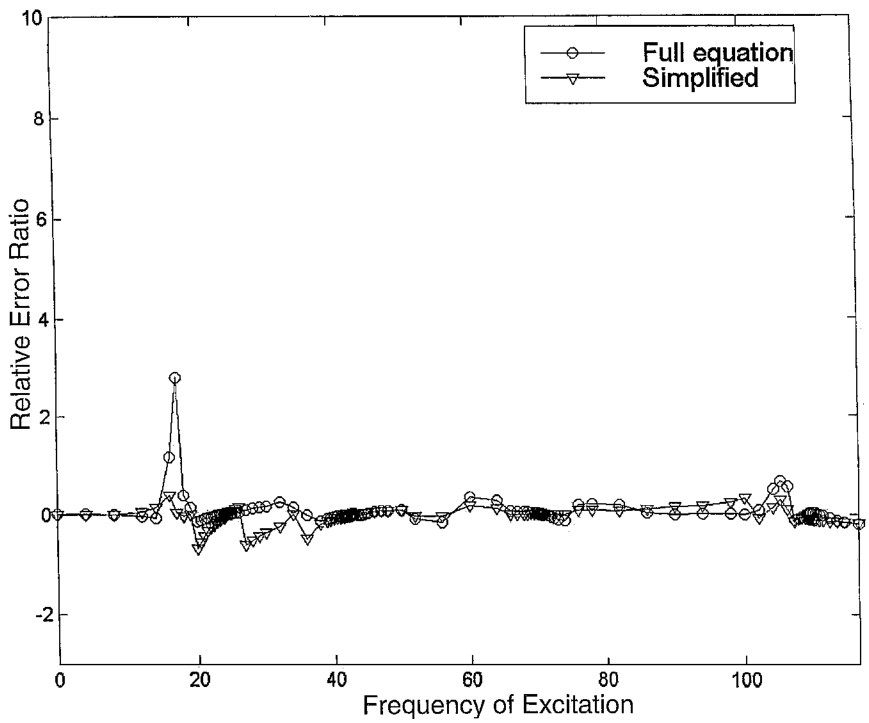

Fig. 6 Accuracy of stress sensitivities obtained by Eqs. (31) and (32) and the simplified Eq. (35).

Finally, a comparison of accuracy of stress sensitivities obtained by the full equations (31) and (32) and the simplified equation (35) is shown in Fig. 6. The accuracy obtained by the simplified equation, as the figure shows, is quite acceptable for optimization purposes.

Conclusions

Preliminary results presented in this paper suggest that a second-order approximate method (based on direct mode displacement and adjoint Ritz-vector-based approximations) can lead to accurate stress frequency-response predictions using very small-order mathematical models. Accuracy of the second-order approximation is less uniform in the case of frequency-dependent dynamic stress sensitivities, but the largest errors in this case seem to concentrate around resonant frequencies and frequencies at which the reduced-order basis used for the adjoint is switched from one frequency band to the next one. A continued search for proper reduced-order bases for the adjoint will hopefully lead to improvements of stress derivative accuracy around those frequencies. In all stress and stress-sensitivity second-order approximations a frequency-dependent adjoint Ritz-vector reduced basis led to more accurate results compared with a fixed static Ritz-vector reduced basis. Accuracy of the reduced-order adjoint approximations themselves was found to be very poor, and this problem should be studied further.

The calculation of the frequency response of large-scale structural dynamic systems at a large number of excitation frequencies, including response sensitivities, is a formidable task. The second-order approximation with the right selection of direct and adjoint reduced-order bases offers to significantly reduce the computational resources required to tackle this problem, by replacing it with a series of approximate direct and adjoint problems that are very small and computationally "cheap."

Appendix: Derivation of a Second-Order Approximation Formula

The second-order approximation of Eq. (20) is obtained in a process similar to that used for obtaining central-difference second-order derivative approximations. Alternative expressions for evaluating the desired response function are manipulated to eliminate the first-order error terms. Let the direct and adjoint solutions of a linear matrix equation be

$$[A]\{x\} = \{b\} \tag{A1}$$

$$[A]^T\{\eta\} = \{c\} \tag{A2}$$

with the response y calculated from either the direct solution $\{x\}$ or the adjoint solution $\{\eta\}$ by

$$y = \{c\}^T\{x\} = \{c\}^T[A]^{-1}\{b\} = \{\eta\}^T\{b\} \tag{A3}$$

An error $\{\delta x\}$ in the direct solution will lead to a corresponding error in the calculated response

$$\delta y_1 = \{c\}^T\{\delta x\} \tag{A4}$$

and an error $\{\delta\eta\}$ in the adjoint solution will lead to a corresponding error in the calculated response

$$\delta y_2 = \{\delta\eta\}^T\{b\} \tag{A5}$$

The assumption here is that the inexact direct or adjoint solutions of Eq. (A1) or (A2) are used to evaluate the response together with the exact $\{b\}$ and $\{c\}$ vectors.

If there is no error in Eqs. (A1) and (A2) and their solutions $\{x\}$ and $\{\eta\}$, then the $\{b\}$ and $\{c\}$ vectors in Eqs. (A4) and (A5) can be replaced as follows:

$$\delta y_1 = \{\eta\}^T[A]\{\delta x\} \tag{A6}$$

$$\delta y_2 = \{\delta\eta\}^T[A]\{x\} \tag{A7}$$

When these two equations are added, the results naturally lead to

$$\begin{aligned}\delta y_1 + \delta y_2 &= \{\eta\}^T[A]\{\delta x\} + \{\delta\eta\}^T[A]\{x\} \\ &= \{\eta + \delta\eta\}^T[A]\{x + \delta x\} - \{\eta\}^T[A]\{x\} - \{\delta\eta\}^T[A]\{\delta x\}\end{aligned} \tag{A8}$$

The first two terms on the right-hand side will immediately be recognized as the error in the expression $\{\eta\}^T[A]\{x\}$ when the exact $\{x\}$ and $\{\eta\}$ are replaced by approximate $\{x\} + \{\delta x\}$ and $\{\eta\} + \{\delta\eta\}$. Thus, based on Eqs. (A4–A8)

$$\delta(\{c\}^T\{x\}) + \delta(\{\eta\}^T\{b\}) - \delta(\{\eta\}^T[A]\{x\}) = -\{\delta\eta\}^T[A]\{\delta x\} \tag{A9}$$

Equation (A9) is recognized as the error in the expression

$$\{c\}^T\{x\} + \{\eta\}^T\{b\} - \{\eta\}^T[A]\{x\} \tag{A10}$$

when the exact direct and adjoint solutions are replaced by approximate ones. The expression (A10), when exact direct and adjoint solutions are used together with exact $[A]$, $\{b\}$, $\{c\}$, is just the response y because with exact solutions Eqs. (A1) or (A2) can be substituted into Eq. (A10) to cancel either $\{\eta\}^T\{b\} - \{\eta\}^T[A]\{x\}$ or $\{c\}^T\{x\} - \{\eta\}^T[A]\{x\}$.

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